$Matrices\ and\ their\ Applications$

$Matrix: is\ an\ array\ contains\ of\ rows\ and\ columns$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots. & \ldots\ldots.. & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & \ldots\ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & \ldots.. & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots. & \ldots. & a_{mn} \end{bmatrix}$$

$$A_{m \times n} \quad m = No.\ of\ rows,\ and\ \ n = No.\ of\ Cols.$$

$$A = \left[a_{ij}\right]\ i = No.\ of\ rows,\ and\ \ j = No.\ of\ Cols.$$

Types of Matrices

1 – Square Matrix: $m = n$ or $i = j$

2 – Diagonal Matrix: is a square matrix, all elements are equal zero except elements of the main diagonal $[a_{ij} = 0$ in all cases $i \neq j]$

3 – Identity Matrix: is a square matrix, which elements of the main diagonal are one, and other elements are zero, denoted by I.

4 – Symmetric Matrix: $[a_{ij} = a_{ji}]$

5 – Sub – Matrix

$$The\ Null\ Matrix$$

$$The\ Upper-Triangular\ Matrix$$

$$The\ Lower-Triangular\ Matrix$$

$$The\ Upper-Unit-Triangular\ Matrix$$

Operations on Matrices

1− *Addition and Subtraction*

$A_{n\times m} \pm B_{k\times l}$ *o.k. if and only if* $n=k$ *and* $m=l$

2 − *Multiply by constant and divide on constant*

Ex:

$$2\times\begin{bmatrix}4 & 0\\ -1 & 5\\ 6 & 1\end{bmatrix}=\begin{bmatrix}8 & 0\\ -2 & 10\\ 12 & 2\end{bmatrix}$$

$$\frac{1}{2}\times\begin{bmatrix}4 & 0\\ -6 & 8\\ 6 & 5\end{bmatrix}=\begin{bmatrix}2 & 0\\ -3 & 4\\ 3 & \frac{5}{2}\end{bmatrix}$$

Determinant: is the value of a square matrix.

A determinant of first order consists of a single element like a and its value equal a.

Ex: Det. of $[20] = |20| = 20$

A determinant of second order consists of $2^2 = 4$ elements

Ex: $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ Det. of $A = a_1 b_2 - a_2 b_1$

Third order

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$A = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Note

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$Example$$

$$A = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$$*\ Use\ \ first\ \ row:$$

$$A = 1\begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 3 & 3 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ 3 & 2 \end{vmatrix}$$

$$= 1(-6-2) - 2(0-3) - 3(0+6)$$

$$= -8 + 6 - 18 = -20$$

$$* \; Use \; second \; row:$$

$$A = -0\begin{vmatrix} 2 & -3 \\ 2 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -3 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$$

$$= 0 - 2(3+9) - 1(2-6)$$

$$= -24 + 4 = -20$$

$$* \; Use \; third \; row:$$

$$A = 3\begin{vmatrix} 2 & -3 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}$$

$$= 3(2-6) - 2(1-0) + 3(-2-0)$$

$$= -12 - 2 - 6 = -20$$

Minor and Co-factors

The minor of an element of a determinant (of one order smaller) lift out on the row and column deleting with through that element.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Minor of } c_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \quad \text{and} \quad \text{Minor of } b_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

Example: Let

$$D = \begin{vmatrix} 1 & -3 & 4 & 5 \\ 3 & 0 & -1 & 6 \\ -4 & -2 & 2 & -6 \\ -5 & 7 & 9 & 8 \end{vmatrix}$$

$$\text{The Minor of } 9 = \begin{vmatrix} 1 & -3 & 5 \\ 3 & 0 & 6 \\ -4 & -2 & -6 \end{vmatrix} = \ldots\ldots\ldots\ldots$$

$The\ Co-factor\ of\ any\ element\ of\ a\ \det er\min ant\ (A_{ij})$

$that\ is\ (-1)^{i+j}\ times\ the\ \min or\ of\ a_{ij}.$

$Co-factor = (-1)^{i+j} \times the \min or\ of\ that\ element.$

$Example:$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$Co-factor\ of\ c_2 = (-1)^{2+3} \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = (-1)(a_1 b_3 - a_3 b_1)$$

$$Other\ method\ to\ evaluate the\ value\ of\ \det er\min ants$$

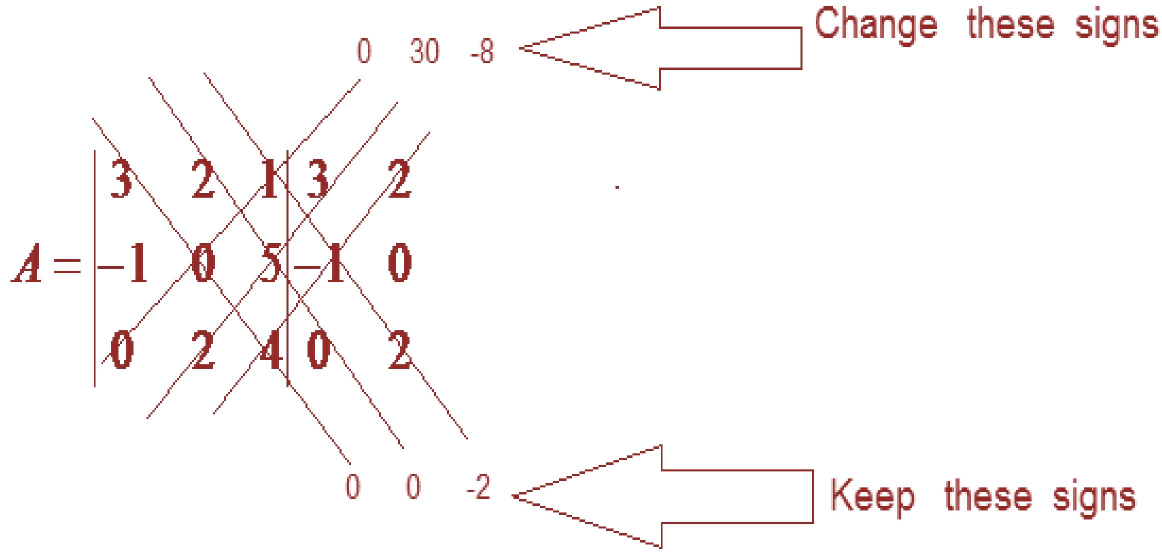
$$Example$$

$$A = \begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 5 \\ 0 & 2 & 4 \end{vmatrix} \begin{matrix} 3 & 2 \\ -1 & 0 \\ 0 & 2 \end{matrix}$$

$$Det.\ of\ \ A = -2 - 22 = -24$$

Other method to evaluate the value of determinants

Example



$$Det.\ of\ \ A = -2 - 22 = -24$$

$\text{Reduction formula for evaluating the determinants}$

$\text{The formula for the determinant of an } (n \times n) \text{ matrix}$

$A = (a_{ij}) \text{ is}$

$$Det.\ A = \left(\frac{1}{a_{11}}\right)^{n-2} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots\cdots & \cdots\cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots\cdots & \cdots\cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

Example:

$$\begin{vmatrix} 1 & 0 & 2 & -1 \\ 3 & -2 & 6 & 4 \\ 5 & 4 & 3 & 0 \\ 2 & 2 & -5 & 6 \end{vmatrix}$$

$$= \left(\frac{1}{1}\right)^{4-2} \begin{vmatrix} \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix} \end{vmatrix}$$

$$= 1\begin{vmatrix} -2 & 0 & 7 \\ 4 & -7 & 5 \\ 2 & -9 & 8 \end{vmatrix}$$

$$= \left(\frac{1}{-2}\right)^{3-2}\begin{vmatrix} \begin{vmatrix} -2 & 0 \\ 4 & -7 \end{vmatrix} & \begin{vmatrix} -2 & 7 \\ 4 & 5 \end{vmatrix} \\ \begin{vmatrix} -2 & 0 \\ 2 & -9 \end{vmatrix} & \begin{vmatrix} -2 & 7 \\ 2 & 8 \end{vmatrix} \end{vmatrix}$$

$$= -\frac{1}{2}\begin{vmatrix} 14 & -38 \\ 18 & -30 \end{vmatrix}$$

$$= -\frac{1}{2}(-420 + 684) = -132$$

Useful Facts about determinants

1- If two rows or (columns) of a matrix are identical, the determinant is zero.

2- Interchanging two rows or (columns) of a matrix, change the sign of its determinant.

3- The determinant of a matrix is the sum of the products of the elements of the i^{th} row (column) by their cofactors, for any i.

4-The determinant of the transpose of a matrix is equal to the original determinant. ("Transpose" means to write the rows as columns).

5- If each element of some row or (column) of a matrix are multiplied by a constant C, the determinant is multiplied by C.

6- If all elements of a matrix above the main diagonal (or all below it) are zero, the determinant of a matrix is [the product of the elements on the main diagonal].

7- If the elements of any row (or column) of a matrix are multiplied by the cofactors of the corresponding elements of a different row (or column) and these products are summed. The sum is zero.

8- If each element of a row (column) of a matrix is multiplied by a constant, C, and the results added to a different row (or column) the determinant is not changed.

Example: Prove without actual expansion that the following determinant vanishes.

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Solution

$$Col.2+Col.3=\begin{vmatrix} 1 & a & b+c+a \\ 1 & b & c+a+b \\ 1 & c & a+b+c \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}=Zero\ (becouse\ Col.1=Col.3)$$

$$Multiplication\ of\ Matrices$$

$$A_{mn} \times B_{kl}\ \ is\ o.k.\ if\ \ and\ only\ if\ \ n = k$$

$$A \times B \neq B \times A$$

$$A_{mn} \times B_{nk} = C_{mk}$$

$$A \times A = A^2\ is\ o.k.\ if\ \ A\ is\ a\ square\ matrix$$

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad and \quad B=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A\times B=\begin{bmatrix} 1a+4b+7c & 2a+5b+8c & 3a+6b+9c \\ 1d+4e+7f & 2d+5e+8f & 3d+6e+9f \\ 1g+4h+7i & 2g+5h+8i & 3g+6h+9i \end{bmatrix}$$

Example:

$$A=\begin{bmatrix}3 & 1 & 2 & 5\\ 1 & 0 & 3 & 1\end{bmatrix} \quad and \quad B=\begin{bmatrix}1 & 3\\ 0 & 5\\ 2 & 6\\ 1 & 0\end{bmatrix}$$

$$A\times B=\begin{bmatrix}3+0+4+5 & 9+5+12+0\\ 1+0+6+1 & 3+0+18+0\end{bmatrix}=\begin{bmatrix}12 & 26\\ 8 & 21\end{bmatrix}$$

$Matrices \ \ Multiplication \ has \ the \ \ following \ \ properties:$

$$(AB)C = A(BC) \qquad (Associative \ \ Law)$$

$$A(B + C) = AB + AC \qquad (Left \ \ distribution \ \ Law)$$

$$(A + B)C = AC + BC \qquad (Right \ \ distribution \ \ Law)$$

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

Inverse of Square Matrices

$$If\ \ M_{n\times n}\times P_{n\times n}=P_{n\times n}\times M_{n\times n}=I$$

$$We\ call\ \ P\ \ the\ inverse\ of\ \ M,\quad P=M^{-1}$$

To find the inverse of a matrix whose determinant is not zero, (Det. formula)

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$1-\textit{Construct the matrix of cofactors of } A:$$

$$Cof.\,A=\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

Such that:

$$A_1=(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$B_1=(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$C_1=(-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$2-\text{Construct the transpose matrix of cofactors,}$

$(\text{called the adjoint of } A)$

$$Adj.\,A = (Cof.\,A)^T = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$3-\text{Then}$

$$A^{-1} = \frac{1}{Det.\,A} \times Adj.\,A$$

Example: Use the determinant formula to find the inverse of the following matrix:

$$A=\begin{bmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{bmatrix}$$

Solution:

$$Cof.\,A=\begin{bmatrix} +\begin{vmatrix}2 & 3\\ -1 & -1\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 3 & -1\end{vmatrix} & +\begin{vmatrix}1 & 2\\ 3 & -1\end{vmatrix}\\ -\begin{vmatrix}3 & -4\\ -1 & -1\end{vmatrix} & +\begin{vmatrix}2 & -4\\ 3 & -1\end{vmatrix} & -\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix}\\ +\begin{vmatrix}3 & -4\\ 2 & 3\end{vmatrix} & -\begin{vmatrix}2 & -4\\ 1 & 3\end{vmatrix} & +\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}\end{bmatrix}$$

$$Cof.A=\begin{bmatrix}1 & 10 & -7\\ 7 & 10 & 11\\ 17 & -10 & 1\end{bmatrix}$$

$$Adj.A=\begin{bmatrix}1 & 7 & 17\\ 10 & 10 & -10\\ -7 & 11 & 1\end{bmatrix}$$

$$Det.\ A=60$$

$$\therefore A^{-1}=\frac{1}{60}\begin{bmatrix}1 & 7 & 17\\ 10 & 10 & -10\\ -7 & 11 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{60} & \frac{7}{60} & \frac{17}{60}\\ \frac{10}{60} & \frac{10}{60} & \frac{-10}{60}\\ \frac{-7}{60} & \frac{11}{60} & \frac{1}{60}\end{bmatrix}$$

$Simultaneous\ linear\ algebraic\ equations:$

$The\ general\ forms\ of\ these\ type\ of\ equations\ are:$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots\dots\dots\dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots\dots\dots\dots a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots\dots\dots\dots\dots a_{3n}x_n = b_3$$

$$\vdots$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots\dots\dots\dots\dots a_{nn}x_n = b_n$$

$$There\ are\ (n)\ equations\ and\ (n)\ unknowns;$$

$$(x_1,\ x_2,\ x_3\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ x_n)$$

$$It\ can\ be\ written\ [Ax = B]$$

$$Such\ that;$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad and \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Methods of Solution

1 – *Cramer's Rule*

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$Let \quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

This method is used when $D \neq 0$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

$If \quad D = 0, \; Donot\, use\, this \;\; method$

$$2-Gauss\ method\ (Gauss\ elimination,\ or\ Gauss\ reduction)$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots\ldots a_{1n}x_n = b_1 \quad \ldots\ldots\ldots\ldots(1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots\ldots a_{2n}x_n = b_2 \quad \ldots\ldots\ldots\ldots(2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots\ldots\ldots\ldots a_{3n}x_n = b_3 \quad \ldots\ldots\ldots\ldots(3)$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots\ldots\ldots a_{nn}x_n = b_n \quad \ldots\ldots\ldots\ldots(n)$$

$$The\ equations\ reduced\ to\ upper\ triangular\ matrix$$

Procedure:

1 – *Normalization*

2 – *Divide* $Eq.(1)$ *by* a_{11} *to get* (x_1), *after normalization.*

3 – *Eliminate* (x_1) *from* $Eq.(2)$ *to* $Eq.(n)$.

4 – *Repeat by dividing* $Eq.(2)$ *by new* (a_{22}) *to get* (x_2)

5 – *From bottom get* $(x_n, x_{n-1}, \ldots\ldots\ldots, x_1)$ *(back substitution)*

$$OR: \ Gauss\ method$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \ \ \ldots\ldots\ldots\ldots(1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \ \ \ldots\ldots\ldots\ldots(2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \ \ \ldots\ldots\ldots\ldots(3)$$

From Eqs. (1) & (2) *and from Eqs.* (1) & (3), *eliminate* x_1 *from Eqs.* (2) & (3).

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \ \ \ldots\ldots\ldots\ldots(1a)$$

$$c_{22}x_2 + c_{23}x_3 = d_2 \ \ \ldots\ldots\ldots\ldots(2a)$$

$$c_{32}x_2 + c_{33}x_3 = d_3 \ \ \ldots\ldots\ldots\ldots(3a)$$

$From\ Eqs.\ (2a)\ \&\ (3a)\ ,\ eliminate\ \ x_2\ \ from\ \ Eq.\ \ (3a).$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \ \ ....(1b)$$

$$c_{22}x_2 + c_{23}x_3 = d_2 \ \ ....(2b)$$

$$e_{33}x_3 = f_3 \ \ ....(3b)$$

$From\ \ Eq.\ (3b),\ \ get\ \ the\ \ value\ \ of\ \ \ x_3$

$From\ \ Eq.\ (2b),\ \ get\ \ the\ \ value\ \ of\ \ \ x_2\ \ after\ \ Subst.\ the\ value\ of\ \ x_3$

$From\ \ Eq.\ (1b),\ \ get\ \ the\ \ value\ \ of\ \ \ x_1\ \ after\ \ Subst.\ the\ value\ of\ \ x_3\ \&\ x_2$

$3-Inverse\ \ matrix\ method:$

$$Ax = B$$

$$AA^{-1}x = A^{-1}B$$

$$Ix = A^{-1}B$$

$$\therefore x = A^{-1}B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad and \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Example: Solve the system of these equations using:

(1) *Cramer's rule*, (2) *Gauss method and* (3) *Invese matrix method.*

$$2x+3y-4z=-3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x+2y+3z=3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$3x-y-z=6 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Solution

(1) *Cramer's rule,*

$$D=\begin{vmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{vmatrix}=60$$

$$x=\frac{\begin{vmatrix}-3 & 3 & -4\\ 3 & 2 & 3\\ 6 & -1 & -1\end{vmatrix}}{60}=\frac{120}{60}=2,\quad y=\frac{\begin{vmatrix}2 & -3 & -4\\ 1 & 3 & 3\\ 3 & 6 & -1\end{vmatrix}}{60}=\frac{-60}{60}=-1$$

$$z=\frac{\begin{vmatrix}2 & 3 & -3\\ 1 & 2 & 3\\ 3 & -1 & 6\end{vmatrix}}{60}=\frac{60}{60}=1$$

$$\therefore\ x=2,\quad y=-1\ \ and\ \ z=1$$

(2) *Gauss method*,

$$x + 2y + 3z = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$2x + 3y - 4z = -3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$3x - y - z = 6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$Eq.(1)$ $$x + 2y + 3z = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

$2Eq.(1) - Eq.(2)$ $$y + 10z = 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

$3Eq.(1) - Eq.(3)$ $$7y + 10z = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

$$x + 2y + 3z = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

$$y + 10z = 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

$7Eq.(2a) - Eq.(3a)$ $$60z = 60 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

$$x + 2y + 3z = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

$$y + 10z = 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

$7Eq.(2a) - Eq.(3a)$ $$60z = 60 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

$$From\ \ Eq.\,(3b)\ \Rightarrow\ z = 1$$

$$From\ Eq.\,(2b)\ \Rightarrow\ y + 10(1) = 9 \Rightarrow\ y = 9 - 10 = -1$$

$$From\ Eq.\,(1b)\ \Rightarrow x + 2(-1) + 3(1) = 3 \Rightarrow\ x = 3 - 3 + 2 = 2$$

$$\therefore\ x = 2,\quad y = -1\ \ and\ \ z = 1$$

$OR, \quad (2)\ Gauss\ method,$

$$\left|\begin{matrix} 1 & 2 & 3 & \vdots & 3 \\ 2 & 3 & -4 & \vdots & -3 \\ 3 & -1 & -1 & \vdots & 6 \end{matrix}\right| \begin{matrix} \\ 2R1-R2 \\ 3R1-R3 \end{matrix}$$

$$\left|\begin{matrix} 1 & 2 & 3 & \vdots & 3 \\ 0 & 1 & 10 & \vdots & 9 \\ 0 & 7 & 10 & \vdots & 3 \end{matrix}\right| \begin{matrix} \\ \\ 7R2-R3 \end{matrix}$$

$$\left|\begin{matrix} 1 & 2 & 3 & \vdots & 3 \\ 0 & 1 & 10 & \vdots & 9 \\ 0 & 0 & 60 & \vdots & 60 \end{matrix}\right| \begin{matrix} x = 3-2(-1)-3(1) = 2 & \\ y = 9-10(1) = -1 & \Uparrow \\ \Rightarrow z = \dfrac{60}{60} = 1 & \Uparrow \end{matrix}$$

$$\therefore x = 2, \quad y = -1 \quad and \quad z = 1$$

(3) *Inverse matrix method*

$$A=\begin{bmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{bmatrix}, \quad B=\begin{bmatrix}-3\\ 3\\ 6\end{bmatrix} \quad and \quad x=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$$Cof.\ A=\begin{bmatrix}1 & 10 & -7\\ 7 & 10 & 11\\ 17 & -10 & 1\end{bmatrix}$$

$$Adj.\ A=\begin{bmatrix}1 & 7 & 17\\ 10 & 10 & -10\\ -7 & 11 & 1\end{bmatrix}$$

$$Det.\ A=\begin{vmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{vmatrix}=60$$

$$\therefore A^{-1} = \frac{1}{60}\begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{60}\begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$$

$$= \frac{1}{60}\begin{bmatrix} -3+21+102 \\ -30+30-60 \\ 21+33+6 \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 120 \\ -60 \\ 60 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore x = 2, \qquad y = -1 \qquad and\ z = 1$$

$4-Iteration\ Method:$

$(i)\ Jacobi\ method$

In this method the equationsare written as:

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{1n}x_n]$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{2n}x_n]$$

$$x_3 = \frac{1}{a_{33}}[b_3 - a_{31}x_1 - a_{32}x_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{3n}x_n]$$

$$\vdots$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}[b_n - a_{n1}x_1 - a_{n2}x_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{n(n-1)}x_{n-1}]$$

$$Strart\ with\ (x_1^{(o)},\ x_2^{(o)},\ \ x_3^{(o)},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x_n^{(o)})$$

$$in\ the\ above\ equationsto\ get\ improved\ estimation$$

$$for\ (x_1^{(1)},\ x_2^{(1)},\ \ x_3^{(1)},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x_n^{(1)})$$

$$and\ repeat\ for\ (k)\ cycles.$$

Example: Solve the following equations:

$$5x_1 + x_2 + 2x_3 = 17 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x_1 + 3x_2 + x_3 = 8 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$2x_1 + x_2 + 6x_3 = 23 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$Solution:$

$1-\ Normalization\ \ (o.k)$

$2-$

$$x_1 = \frac{1}{5}(17 - x_2 - 2x_3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$x_2 = \frac{1}{3}(8 - x_1 - x_3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$x_3 = \frac{1}{6}(23 - 2x_1 - x_2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$3-$

$Start\ \ with\ \ x_1^{(o)} = x_2^{(o)} = x_3^{(o)} = 1$

cycles	0	1	2	3	4	5	6	7
x_1	1	2.8	1.667	2.25	1.88	2.08		
x_2	1	2	0.622	1.255	0.859	1.08		
x_3	1	3.333	2.567	3.174	2.874	3.06		

$$\therefore \; x_1 = 2, \quad x_2 = 1 \quad and \; x_3 = 3$$

$(ii)\ Gauss-Sedal\ method$

This method is the most powerful and most popular method of solution. In fact it is extension to the Jacobi method. The iterative equations are:

$$x_1^{i+1} = \frac{1}{a_{11}}[b_1 - a_{12}x_2^i - a_{13}x_3^i \ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{1n}x_n^i]$$

$$x_2^{i+1} = \frac{1}{a_{22}}[b_2 - a_{21}x_1^{i+1} - a_{23}x_2^i \ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{2n}x_n^i]$$

$$x_3^{i+1} = \frac{1}{a_{33}}[b_3 - a_{31}x_1^{i+1} - a_{32}x_2^{i+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{3n}x_n^i]$$

$$\vdots$$

$$\vdots$$

$$x_n^{i+1} = \frac{1}{a_{nn}}[b_n - a_{n1}x_1^{i+1} - a_{n2}x_2^{i+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_{n(n-1)}^{i+1}x_{n-1}]$$

$$x_1 = \frac{1}{5}(17 - x_2 - 2x_3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$x_2 = \frac{1}{3}(8 - x_1 - x_3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$x_3 = \frac{1}{6}(23 - 2x_1 - x_2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

cycles	0	1	2	3	4	5	6	7
x_1	1	2.8	2.05	1.99				
x_2	1	1.4	1.09	1.01				
x_3	1	2.67	2.97	3.002				

$$\therefore \quad x_1 = 2, \quad x_2 = 1 \quad \text{and} \quad x_3 = 3$$